

FIXED POINTS OF 321-AVOIDING PERMUTATIONS

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ABSTRACT. We describe the distribution of the number and location of the fixed points of permutations that avoid the pattern 321 via a bijection with rooted plane trees on $n + 1$ vertices. Using the local limit theorem for Galton-Watson trees, we are able to give an explicit description of the limit of this distribution.

1. Introduction

For a permutation $\tau \in S_n$ and $\sigma \in S_k$ with $k \leq n$, τ is said to contain an instance of the pattern σ if there exists a subset of indices $i_1 < i_2 < \cdots < i_k$ such that $\tau_{i_s} < \tau_{i_t}$ if and only if $\sigma_s < \sigma_t$ for all $1 \leq s < t \leq k$. The class of permutations that do not contain an instance of σ are called σ -avoiding and $\mathbf{Av}_n(\sigma)$ denotes the set of σ -avoiding permutations in S_n .

In recent years much work has been done in understanding the statistics of various classes of pattern-avoiding permutations. One statistic that has received particular interest is the fixed point statistic [5, 6, 16, 18]. From a combinatorial perspective, much of the work concerning statistics of pattern-avoiding permutation has been in understanding how they are affected by bijections from one pattern-avoiding class to another. Robertson, Saracino, and Zeilberger [18] found a bijection between $\mathbf{Av}_n(321)$ and $\mathbf{Av}_n(132)$ that preserves the fixed point statistic. Elizalde and Pak [5] give another bijection between the same classes that preserves both fixed points and the number of exceedances. From the beginning there has been interest in analyzing the typical values of this statistic. For example, the expected number of fixed points of a uniformly random element of $\mathbf{Av}_n(\sigma)$ is computed in [4] for several cases when σ has length 3.

A number of recent papers [3, 6, 7, 9, 15, 16] have continued this study of pattern-avoiding permutations as random objects. In the context of random permutations, the natural object to consider is the empirical distribution of fixed points. In the authors' previous work in [6] they describe the limiting distribution of the number and location of fixed points for both $\mathbf{Av}_n(123)$ and $\mathbf{Av}_n(231)$ using the global connection with Brownian excursion they established in [7]. In this paper we show that the empirical distribution of fixed points for random permutations in $\mathbf{Av}_n(321)$ is related to another classical probabilistic object, specifically a size-biased Galton-Watson tree.

In [7] the authors also showed that if $\tau_n \in \mathbf{Av}_n(321)$ is uniformly random then

$$\left(\frac{1}{\sqrt{2n}} |\tau_n(\lfloor nt \rfloor) - \lfloor nt \rfloor| \right)_{0 \leq t \leq 1}$$

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converges in distribution to standard Brownian excursion. This describes the structure of the bulk of the permutation, however because of the scaling one cannot extract precise information about τ_n near 1 and n . Moreover, the only information about the fixed points we can obtain from this picture is that they must occur near the beginning or end of the permutation since Brownian excursion is strictly positive except at the end points. To understand the behavior of a permutation τ_n near the beginning and the end, we utilize a bijection with rooted plane trees on $n + 1$ vertices. Rather than take a scaling limit, which would lead back to Brownian excursion, we take a local limit that results in a particular size-biased Galton-Watson tree. The behavior of the permutation τ_n near 1 and n can then be described using the limiting random tree and we use this to give an explicit description of the limiting empirical measure of fixed points. To the best of our knowledge, this is the first use of local limits of random trees in the study of pattern-avoiding permutations and gives another illustration of the value of classical probabilistic limit theorems in the study of random pattern-avoiding permutations.

The bijection we use is a reinterpretation of a bijection found in [5]. The bijection in [5] is from $\mathbf{Av}_n(321)$ to the set of Dyck paths of length $2n$. Under this bijection the fixed points of τ are sent to peaks of height 1. We compose this bijection with the standard bijection from Dyck paths to rooted ordered trees that treats the Dyck path as the contour process of the tree (creating a new edge with upsteps and moving toward the root with a downstep). Under this bijection the peaks of height 1 are sent to leaves connected to the root. In section 2 we make this bijection explicit.

In Section 3 we describe the appropriate limiting object of a sequence of rooted plane trees. Section 4 translates information about this limiting object to information about the number and location of fixed points in $\mathbf{Av}_n(321)$. We can then simplify the description to obtain a clean statement that does not need the formalism of size-biased Galton-Watson trees.

We state our main result, which gives the convergence of the joint distribution of the number and location of the fixed points at the beginning and end of a permutation chosen uniformly from $\mathbf{Av}_n(321)$.

Theorem 1. *Let $((X_i, Y_i))_{i \geq 1}$ be a i.i.d sequence such for each i , X_i and Y_i are independent with*

$$\mathbb{P}(X_i = k) = \mathbb{P}(Y_i = k) = \frac{1}{k2^{2k-1}} \binom{2k-2}{k-1}.$$

Further, let N and M be independent Geometric(1/2) random variables that are also jointly independent of the sequence $((X_i, Y_i))_{i \geq 1}$. If τ_n is a uniformly random element of $\mathbf{Av}_n(321)$ then

$$\left(\sum_{i=1}^{\lfloor n/2 \rfloor} \delta_i \mathbf{1}_{\{\tau_n(i)=i\}}, \sum_{i=\lfloor n/2 \rfloor + 1}^n \delta_{n+1-i} \mathbf{1}_{\{\tau_n(i)=i\}} \right) \xrightarrow{d} \left(\sum_{k=1}^N \delta_{\sum_{j=1}^k X_j} \mathbf{1}_{\{X_k=1\}}, \sum_{k=1}^M \delta_{\sum_{j=1}^k Y_j} \mathbf{1}_{\{Y_k=1\}} \right),$$

the convergence being convergence in distribution on $\mathcal{M}_F(\{1, 2, 3, \dots\})^2$ where $\mathcal{M}_F(\{1, 2, 3, \dots\})$ is the set of finite measures on $\{1, 2, 3, \dots\}$ equipped with the topology of weak convergence.

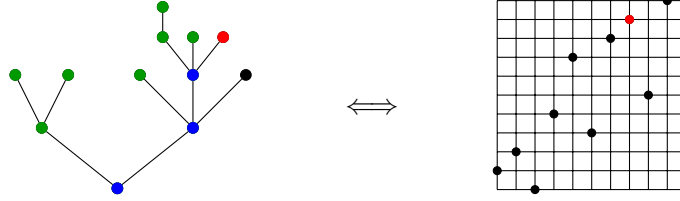
Recall that the topology of weak convergence on $\mathcal{M}_F(\{1, 2, 3, \dots\})$ is the topology generated by the set maps $\{\mu \mapsto \int f d\mu \mid f : \{1, 2, 3, \dots\} \rightarrow \mathbb{R} \text{ is bounded}\}$.

For comparison, the strongest prior result on the location of fixed points for random permutations in $\mathbf{Av}_n(321)$ is the following result from [16].

Theorem 2. [16, Theorem 6.1] *Let τ_n be a uniformly random element of $\mathbf{Av}_n(321)$. For fixed $0 < \epsilon \leq 1/2$ we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n(i) = i \text{ for some } i \in [\epsilon n, (1 - \epsilon)n]) = 0.$$

From our Theorem 1, we immediately obtain the following stronger result.

FIGURE 1. A rooted tree \mathbf{t} and a corresponding permutation τ_n .

Corollary 3. Let τ_n be a uniformly random element of $\mathbf{Av}_n(321)$ and let $0 < a_n \leq b_n$ be sequences of integers. Then

$$\lim_{n \rightarrow \infty} \mathbb{P}(\tau_n(i) = i \text{ for some } i \in [a_n, b_n]) = 0$$

if and only if $a_n \rightarrow \infty$ and $n - b_n \rightarrow \infty$.

If we only wish to know the limiting distribution of the number of fixed points at the beginning and the end of the permutation we have a much simpler solution:

Corollary 4. Let A and B be independent Geometric(2/3) random variables. If τ_n is a uniformly random element of $\mathbf{Av}_n(321)$ then

$$\left(\sum_{i=1}^{\lfloor n/2 \rfloor} \mathbf{1}_{\{\tau_n(i)=i\}}, \sum_{i=\lfloor n/2 \rfloor + 1}^n \mathbf{1}_{\{\tau_n(i)=i\}} \right) \xrightarrow{d} (A, B).$$

The asymptotic distribution of the total number of fixed points can be also be found using the continuity of addition.

Corollary 5. Let Z be distributed as $\mathbb{P}(Z = k) = \frac{4}{9}(k+1)\left(\frac{1}{3}\right)^k$ and let A and B be independent Geometric(2/3) random variables. If τ_n is a uniform random element of $\mathbf{Av}_n(321)$ then

$$\sum_{i=1}^n \mathbf{1}_{\{\tau_n(i)=i\}} \xrightarrow{d} A + B \stackrel{d}{=} Z.$$

We remark that Z follows a NegativeBinomial(2, 1/3) distribution. Although Corollary 5 seems to be new, and we derived from our explicit description of the limiting empirical distribution of fixed points, it can also be found by analyzing the bivariate generating function for fixed points given by [4, Theorem 2.9].

Using one of the many bijections [13, 5, 18] from $\mathbf{Av}_n(321)$ to $\mathbf{Av}_n(132)$ or $\mathbf{Av}_n(213)$ that preserves the number of fixed points we have the following.

Corollary 6. Let Z be distributed like the sum of two independent Geometric(2/3) random variables, i.e. $\mathbb{P}(Z = k) = \frac{4}{9}(k+1)\left(\frac{1}{3}\right)^k$. If π_n is a uniformly random element in $\mathbf{Av}_n(132)$ or $\mathbf{Av}_n(213)$ then

$$\left(\sum_{i=1}^n \mathbf{1}_{\{\pi_n(i)=i\}} \right) \xrightarrow{d} Z.$$

Unlike the results for $\mathbf{Av}_n(321)$, these bijections do not easily give informations about the location of the fixed points for random permutations in $\mathbf{Av}_n(132)$ or $\mathbf{Av}_n(213)$.

2. A bijection between rooted plane trees and $\mathbf{Av}_n(321)$

Let \mathfrak{T}_{n+1} denote the set of rooted plane trees with $n + 1$. We will describe a bijection with rooted plane trees and $\mathbf{Av}_n(321)$ that maps the leaves of the tree to the left-to-right maxima of the corresponding permutation, see Figure 1. This bijection follows from a bijection with $\mathbf{Av}_n(321)$ and Dyck paths of length $2n$ found in [5].

Suppose \mathbf{t} has k leaves and let $\ell(\mathbf{t}) = (l_1, l_2, \dots, l_k)$ be the list of leaves in \mathbf{t} , listed in order of appearance on the depth-first walk of \mathbf{t} . There are two quantities associated with the leaves of the tree needed to construct the bijection. Given the ordering on the vertices of \mathbf{t} given by traversing the exterior of the tree from left to right, starting from the root, let s_i equal the number of vertices in \mathbf{t} less than l_i . Let p_i denote the number of vertices less than l_i in the path from the root to l_i .

Construct two increasing sequences of size k , $A = \{s_i\}_{i=1}^k$ and $B = \{s_i - p_i + 1\}_{i=1}^k$. We define $\tau_{\mathbf{t}}$ pointwise on B

$$\tau_{\mathbf{t}}(s_i - p_i + 1) = s_i$$

for $1 \leq i \leq k$. We then extend $\tau_{\mathbf{t}}$ to $[n] \setminus B$ by assigning values of $[n] \setminus A$ in increasing order.

The resulting function $\tau_{\mathbf{t}}$ is a permutation in $\mathbf{Av}_n(321)$. The set of values for s_i and p_i uniquely determine a rooted plane tree, so we may recover the corresponding rooted tree from the left-to-right maxima of a permutation in $\tau_{\mathbf{t}}$. Let $M = \{m_i\}_{i=1}^k$ denote the set of indices of the left-to-right maxima of $\tau_{\mathbf{t}}$. Then

$$s_i = \tau_{\mathbf{t}}(m_i)$$

and

$$p_i = \tau_{\mathbf{t}}(m_i) - m_i + 1.$$

Finally we note that fixed points in $\tau_{\mathbf{t}}$ correspond to leaves l_i of \mathbf{t} such that $p_i = 1$.

3. The Local Limit Theorem

In this section we introduce the appropriate limit for a sequence of trees $(\mathbf{t}_n)_{n \geq 1}$ such that $\mathbf{t}_n \in \mathfrak{T}_n$. For this we use Neveu's formalism for rooted ordered trees [17]. This will give us a consistent way to refer to the vertices of different trees. We let

$$\mathcal{U} = \bigcup_{n \geq 0} \{1, 2, \dots\}^n,$$

where $\{1, 2, \dots\}^0 = \{\emptyset\}$. For $u \in \mathcal{U}$, we let $|u|$ be the length of u . That is, $|u|$ is the unique integer such that $u \in \{1, 2, \dots\}^{|u|}$. We treat the elements of \mathcal{U} as lists, and for $u, v \in \mathcal{U}$, we denote the concatenation of u and v by uv . The empty list is the identity for this operation, that is, if $u = \emptyset$ then $uv = vu = v$. The set of ancestors of u is defined to be

$$A_u = \{v \in \mathcal{U} : \exists w \in \mathcal{U} \text{ such that } u = vw\}.$$

We can consider \mathcal{U} as the vertex set of a rooted ordered tree U_{∞} , called the Ulam-Harris tree, where we take the lexicographic order on \mathcal{U} as the order, \emptyset as the root, and for every $u \in \mathcal{U}$ and $i \in \{1, 2, 3, \dots\}$ there is an edge from u to ui . By a rooted, ordered tree we will mean a (possibly infinite) subtree \mathbf{t} of U_{∞} such that

- (a) $\emptyset \in \mathbf{t}$.
- (b) If $u \in \mathbf{t}$ then $A_u \subseteq \mathbf{t}$.
- (c) For every $u \in \mathbf{t}$, if $ui \in \mathbf{t}$ then $uj \in \mathbf{t}$ for all $j \leq i$.

A tree $\mathbf{t} \subseteq U_{\infty}$ is considered as rooted and ordered by taking \emptyset to be the root and using the order on \mathbf{t} induced by the lexicographic order on \mathcal{U} . If $u \in \mathbf{t}$, the vertices in A_u are the vertices in

\mathbf{t} on the path from the root to u . For a tree \mathbf{t} and $u \in \mathbf{t}$, define $d_{\mathbf{t}}(u) = 0 \vee \sup\{i : ui \in \mathbf{t}\}$. We also define $d_{\mathbf{t}}(u) = -1$ if $u \notin \mathbf{t}$. A tree \mathbf{t} is called *locally finite* if $d_{\mathbf{t}}(u) < \infty$ for all $u \in \mathcal{U}$ and define

$$\mathfrak{T} = \{\mathbf{t} \subseteq U_{\infty} : \mathbf{t} \text{ is locally finite}\}.$$

We topologize \mathfrak{T} with the convergence $\mathbf{t}_n \rightarrow \mathbf{t}$ if $d_{\mathbf{t}_n}(u) \rightarrow d_{\mathbf{t}}(u)$ for all $u \in \mathcal{U}$. Since \mathcal{U} is countable this convergence can be metrized by, for example, the complete metric

$$D(\mathbf{t}, \mathbf{s}) = \sum_{i=1}^{\infty} \frac{|d_{\mathbf{t}}(f(i)) - d_{\mathbf{s}}(f(i))| \wedge 1}{2^i},$$

where $f : \{1, 2, \dots\} \rightarrow \mathcal{U}$ is a bijection.

Let ξ be a probability distribution on $\{0, 1, 2, \dots\}$ with mean 1, ie. $\sum_i i\xi(i) = 1$. A \mathfrak{T} -valued random variable is called a Galton-Watson tree with offspring distribution ξ if

$$\mathbb{P}(T = \mathbf{t}) = \prod_{u \in \mathbf{t}} \xi(d_{\mathbf{t}}(u))$$

for all $\mathbf{t} \in \mathfrak{T}$. For $\mathbf{t} \in \mathfrak{T}$, let $\mathbf{t}^{[k]}$ be the subtree of \mathbf{t} comprised of vertices of height at most k . Let T be a ξ -Galton-Watson tree and define the probability measure ν_{ξ} on \mathfrak{T} to be the unique measure on infinite trees such that for every k and $\mathbf{t}_0 \in \mathfrak{T}$ with height k ,

$$\nu_{\xi}(\{\mathbf{t} \in \mathfrak{T} : \mathbf{t}^{[k]} = \mathbf{t}_0\}) = (\#_k \mathbf{t}_0) \mathbb{P}(T^{[k]} = \mathbf{t}_0),$$

where $\#_k \mathbf{t}$ is the number of vertices in \mathbf{t} with height k . See e.g. [12, Lemma 1.14] for the existence and uniqueness of this measure. Let \tilde{T} have distribution ν_{ξ} . \tilde{T} is called a size-biased ξ -Galton-Watson tree. The size biased distribution of ξ is the distribution

$$\tilde{\xi}(n) = n\xi(n).$$

Note that this is a probability distribution because ξ has mean 1. Observe that, directly from the definition of ν_{ξ} , we have

$$\mathbb{P}(\#_1 \tilde{T} = k) = \tilde{\xi}(k),$$

which justifies calling \tilde{T} the size-biased tree. For trees $\mathbf{t}_1, \dots, \mathbf{t}_k$, let $\langle \mathbf{t}_1, \dots, \mathbf{t}_k \rangle$ be the rooted ordered tree obtained by attaching the roots of $\mathbf{t}_1, \dots, \mathbf{t}_k$ to a new vertex and calling that vertex the root. Formally, for $\mathbf{t} \in \mathfrak{T}$ and $i \in \{1, 2, \dots\}$, we let $i\mathbf{t} = \{iv \in \mathcal{U} : v \in \mathbf{t}\}$ and define $\langle \mathbf{t}_1, \dots, \mathbf{t}_k \rangle$ as the rooted, ordered tree with vertex set

$$\{\emptyset\} \cup \bigcup_{i=1}^k i\mathbf{t}_i.$$

For our purposes, \tilde{T} has two important properties, which we summarize in the next two lemmas.

Lemma 3.1 ([12, Lemma 2.2]). *Let T_1, T_2, \dots be an i.i.d sequence of ξ -Galton-Watson trees, let \tilde{T} be an independent size-biased ξ -Galton-Watson tree and let (\tilde{X}, N) be independent of $\tilde{T}, T_1, T_2, \dots$ and distributed such that \tilde{X} has distribution $\tilde{\xi}$ and conditionally give $\tilde{X} = k$, N is uniformly distributed on $\{1, 2, \dots, k\}$. Then \tilde{T} is distributed like*

$$\tilde{T} \stackrel{d}{=} \langle T_1, \dots, T_{N-1}, \tilde{T}, T_{N+1}, \dots, T_{\tilde{X}} \rangle.$$

See also the discussions in Section 2 of [14] and Section 5 of [8] for a constructions of \tilde{T} from which the lemma follows immediately. For $\mathbf{t} \in \mathfrak{T}$, let $\#\mathbf{t}$ be the number of vertices of \mathbf{t} .

Lemma 3.2. *Let T be a ξ -Galton-Watson tree and for n such that $\mathbb{P}(\#T = n) > 0$, let T_n be distributed like T conditioned on $\mathbb{P}(\#T = n)$. Then*

$$T_n \xrightarrow{d} \tilde{T}.$$

That is, for every tree $\mathbf{t} \in \mathfrak{T}$ and $k \in \{1, 2, 3, \dots\}$ we have

$$\lim_{n \rightarrow \infty} \mathbb{P}(T_n^{[k]} = \mathbf{t}) = \lim_{n \rightarrow \infty} \mathbb{P}(T^{[k]} = \mathbf{t} \mid \#T = n) = \mathbb{P}(\tilde{T}^{[k]} = \mathbf{t}),$$

where the limit is understood to be along n such that $\mathbb{P}(\#T = n) > 0$.

This lemma has a long history. The case when ξ has finite variance, which is the only case we will need, was proven implicitly in [11] and first stated explicitly in [2]. A version for conditioning on $\#T \geq n$ rather than $\#T = n$ was given in [12]. See [8, Theorem 7.1] and [1] for modern approaches to the general result.

4. Application to fixed points

The connection between local limit theorems for Galton-Watson trees and fixed points of 321-avoiding permutations comes from the following result. For $n \in \{0, 1, 2, \dots\}$ define $\xi(n) = 2^{-n-1}$, i.e., ξ has the Geometric(1/2) distribution. If T is a ξ -Galton-Watson tree and T_n is distributed like T conditioned on $\#T = n$, then T_n is a uniformly random rooted ordered tree with n vertices. Thus, $\tau_n := \tau_{T_{n+1}}$ is a uniformly random element of $\mathbf{Av}_n(321)$.

From the local limit theorem for Galton-Watson trees, we obtain the following result.

Theorem 7. *Let $\mathcal{T}_1, \mathcal{T}_2, \dots$ be an i.i.d. sequence of Geometric(1/2)-Galton-Watson trees, let (\tilde{X}, N) be independent of $\mathcal{T}_1, \mathcal{T}_2, \dots$ and distribution such that \tilde{X} has a size-biased Geometric(1/2) distribution and conditionally give $\tilde{X} = k$, N is uniformly distributed on $\{1, 2, \dots, k\}$. If τ_n is a uniformly random element of $\mathbf{Av}_n(321)$ then*

$$\left(\sum_{i=1}^{\lfloor n/2 \rfloor} \delta_i \mathbf{1}_{\{\tau_n(i)=i\}}, \sum_{i=\lfloor n/2 \rfloor + 1}^n \delta_{n+1-i} \mathbf{1}_{\{\tau_n(i)=i\}} \right) \xrightarrow{d} \left(\sum_{i=1}^{N-1} \delta_{\sum_{j=1}^i \# \mathcal{T}_j} \mathbf{1}_{\{\# \mathcal{T}_i=1\}}, \sum_{i=N+1}^{\tilde{X}} \delta_{\sum_{j=i}^{\tilde{X}} \# \mathcal{T}_j} \mathbf{1}_{\{\# \mathcal{T}_i=1\}} \right),$$

the convergence being convergence in distribution on $\mathcal{M}_F(\{1, 2, 3, \dots\})^2$ where $\mathcal{M}_F(\{1, 2, 3, \dots\})$ is the set of finite measures on $\{1, 2, 3, \dots\}$ equipped with the topology of weak convergence.

Proof. Let T_{n+1} be distributed like \mathcal{T}_1 conditioned on $\#\mathcal{T}_1 = n+1$. Since $T_n \rightarrow_d \tilde{T}$ by Lemma 3.2 and \mathfrak{T} admits a complete metric, we may use the Skorokhod Representation Theorem (see e.g. [10, Theorem 4.30]) to construct a version of the sequence $(T_n)_{n \geq 0}$ and \tilde{T} such that $T_n \rightarrow \tilde{T}$ almost surely. For $\mathbf{t} \in \mathfrak{T}$ and $1 \leq i \leq d_{\mathbf{t}}(\emptyset)$, define $F_i^{\mathbf{t}} \in \mathfrak{T}$ to be the fringe subtree of \mathbf{t} above i . That is, $F_i^{\mathbf{t}} \in \mathfrak{T}$ is the rooted ordered tree with vertex set $\{u \in \mathcal{U} : iu \in \mathbf{t}\}$. In particular, note that $\mathbf{t} = \langle F_1^{\mathbf{t}}, \dots, F_{d_{\mathbf{t}}(\emptyset)}^{\mathbf{t}} \rangle$.

Define

$$N_n = \inf \left\{ k : \sum_{i=1}^k \#F_k^{T_n} > \lfloor n/2 \rfloor \right\} \quad \text{and} \quad N = \inf \left\{ k : \sum_{i=1}^k \#F_k^{\tilde{T}} = \infty \right\}.$$

By Lemma 3.1, N is the unique index such that $\#F_N^{\tilde{T}} = \infty$. Letting $\tau_n = \tau_{T_{n+1}}$, we observe that

$$\begin{aligned} & \left(\sum_{i=1}^{\lfloor n/2 \rfloor} \delta_i \mathbf{1}_{\{\tau_n(i)=i\}}, \sum_{i=\lfloor n/2 \rfloor+1}^n \delta_{n+1-i} \mathbf{1}_{\{\tau_n(i)=i\}} \right) \\ &= \left(\sum_{i=1}^{N_{n+1}-1} \delta_{\sum_{j=1}^i \#F_j^{T_{n+1}}} \mathbf{1}_{\{\#F_i^{T_{n+1}}=1\}}, \sum_{i=N_{n+1}}^{d_{T_{n+1}}(\emptyset)} \delta_{\sum_{j=i}^{d_{T_{n+1}}(\emptyset)} \#F_j^{T_{n+1}}} \mathbf{1}_{\{\#F_i^{T_{n+1}}=1\}} \right), \end{aligned}$$

Since $T_n \rightarrow T$ almost surely there exists $M = M(\omega)$ such that for all $n \geq M(\omega)$, $d_{T_n}(\emptyset) = d_{\tilde{T}}(\emptyset)$, $F_j^{T_n} = F_j^{\tilde{T}}$ for all $1 \leq j \leq d_{\tilde{T}}(\emptyset)$ such that $\#F_j^{\tilde{T}} < \infty$, $N_n = N$, and $\#F_{N_n}^{T_n} \geq 3n/4$. Therefore, for $n \geq M(\omega)$

$$\begin{aligned} & \left(\sum_{i=1}^{N_{n+1}-1} \delta_{\sum_{j=1}^i \#F_j^{T_{n+1}}} \mathbf{1}_{\{\#F_i^{T_{n+1}}=1\}}, \sum_{i=N_{n+1}}^{d_{T_{n+1}}(\emptyset)} \delta_{\sum_{j=i}^{d_{T_{n+1}}(\emptyset)} \#F_j^{T_{n+1}}} \mathbf{1}_{\{\#F_i^{T_{n+1}}=1\}} \right) \\ &= \left(\sum_{i=1}^{N-1} \delta_{\sum_{j=1}^i \#F_j^{\tilde{T}}} \mathbf{1}_{\{\#F_i^{\tilde{T}}=1\}}, \sum_{i=N+1}^{d_{\tilde{T}}(\emptyset)} \delta_{\sum_{j=i}^{d_{\tilde{T}}(\emptyset)} \#F_j^{\tilde{T}}} \mathbf{1}_{\{\#F_i^{\tilde{T}}=1\}} \right). \end{aligned}$$

Noting that

$$\begin{aligned} & \left(\sum_{i=1}^{N-1} \delta_{\sum_{j=1}^i \#F_j^{\tilde{T}}} \mathbf{1}_{\{\#F_i^{\tilde{T}}=1\}}, \sum_{i=N+1}^{d_{\tilde{T}}(\emptyset)} \delta_{\sum_{j=i}^{d_{\tilde{T}}(\emptyset)} \#F_j^{\tilde{T}}} \mathbf{1}_{\{\#F_i^{\tilde{T}}=1\}} \right) \\ &= {}_d \left(\sum_{i=1}^{N-1} \delta_{\sum_{j=1}^i \#T_j} \mathbf{1}_{\{\#T_i=1\}}, \sum_{i=N+1}^{\tilde{X}} \delta_{\sum_{j=i}^{\tilde{X}} \#T_j} \mathbf{1}_{\{\#T_i=1\}} \right), \end{aligned}$$

completes the proof. ■

Using the special properties of the geometric distribution, we are able to simplify the statement of the theorem and, indeed, remove the trees from the picture entirely.

Proof of Theorem 1. Since $\mathbb{P}(\tilde{X} = j) = j(1/2)^{j+1}$ we compute that for $k \geq 0$

$$\mathbb{P}(N = k+1) = \sum_{j=k+1}^{\infty} \frac{1}{j} \mathbb{P}(\tilde{X} = j) = \sum_{j=k+1}^{\infty} \frac{1}{2^{j+1}} = \frac{1}{2^{k+1}}.$$

Thus $N-1$ has a Geometric($1/2$) distribution. Similarly, for $k \geq 0$ we have

$$\mathbb{P}(\tilde{X} - N = k) = \sum_{j=k+1}^{\infty} \mathbb{P}(\tilde{X} - N = k, \tilde{X} = j) = \sum_{j=k+1}^{\infty} \frac{1}{j} \mathbb{P}(\tilde{X} = j) = \frac{1}{2^{k+1}},$$

which shows that $\tilde{X} - N$ has a Geometric($1/2$) distribution. Moreover for $k, j \geq 0$,

$$\begin{aligned} \mathbb{P}(N-1 = k, \tilde{X} - N = j) &= \mathbb{P}(N = k+1, \tilde{X} = j+k+1) = \frac{1}{j+k+1} \mathbb{P}(\tilde{X} = j+k+1) \\ &= \frac{1}{j+k+1} (j+k+1) \frac{1}{2^{j+k+2}} \\ &= \mathbb{P}(N-1 = k) \mathbb{P}(\tilde{X} - N = j). \end{aligned}$$

Thus $N - 1$ and $\tilde{X} - N$ are independent Geometric($1/2$) random variables. Furthermore, for $\mathbf{t} \in \mathfrak{T}$, we have that

$$\mathbb{P}(T_1 = \mathbf{t}) = \prod_{v \in \mathbf{t}} \frac{1}{2^{d(v)+1}} = \frac{1}{2^{2\#\mathbf{t}-1}},$$

from which it follows that

$$\mathbb{P}(\#T_1 = k) = \frac{1}{2^{2k-1}} C_{k-1} = \frac{1}{k 2^{2k-1}} \binom{2k-2}{k-1}$$

where $C_{k-1} = \frac{1}{k} \binom{2k-2}{k-1}$ is the $(k-1)$ 'st Catalan number. ■

Proof of Corollary 4. Let $Y_1 = \sum_{k=1}^N \mathbf{1}_{\{X_k=1\}}$ and $Y_2 = \sum_{k=1}^M \mathbf{1}_{\{Y_k=1\}}$. By Theorem 1, the limiting distribution of the total number of fixed points up to $\lfloor n/2 \rfloor$ is counted by Y_1 and the total after is counted by Y_2 . Both are independent copies of the same random variable Y with distribution

$$\begin{aligned} \mathbb{P}(Y = k) &= \sum_{t=0}^{\infty} \mathbb{P}(N = t) \binom{t}{k} 2^{-t} \\ &= \sum_{t=0}^{\infty} 2^{-2t-1} \binom{t}{k} \\ &= 2 * 3^{-k-1} \end{aligned}$$

which is the probability mass function for a Geometric($2/3$) random variable. ■

References

- [1] Romain Abraham and Jean-François Delmas. Local limits of conditioned Galton-Watson trees: the infinite spine case. *Electron. J. Probab.*, 19:no. 2, 19, 2014.
- [2] David Aldous and Jim Pitman. Tree-valued Markov chains derived from Galton-Watson processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 34(5):637–686, 1998.
- [3] Frédérique Bassino, Mathilde Bouvel, Valentin Féray, Lucas Gerin, and Adeline Pierrot. The Brownian limit of separable permutations. *arXiv preprint arXiv:1602.04960*, 2016.
- [4] Sergi Elizalde. *Consecutive patterns and statistics on restricted permutations*. PhD thesis, Ph. D. thesis, Universitat Politècnica de Catalunya, 2004.
- [5] Sergi Elizalde and Igor Pak. Bijections for refined restricted permutations. *Journal of Combinatorial Theory, Series A*, 105:207–219, 2004.
- [6] Christopher Hoffman, Douglas Rizzolo, and Erik Slivken. Pattern-avoiding permutations and Brownian excursion, Part II: Fixed points. *arXiv preprint arXiv:1506.04174*, 2015.
- [7] Christopher Hoffman, Douglas Rizzolo, and Erik Slivken. Pattern-avoiding permutations and Brownian excursion Part I: Shapes and fluctuations. *Random Structures and Algorithms [accepted]*, 2016.
- [8] Svante Janson. Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation. *Probab. Surv.*, 9:103–252, 2012.
- [9] Svante Janson. Patterns in random permutations avoiding the pattern 132. *arXiv preprint arXiv:1401.5679*, 2014.
- [10] Olav Kallenberg. *Foundations of modern probability*. Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002.
- [11] Douglas P. Kennedy. The Galton-Watson process conditioned on the total progeny. *J. Appl. Probability*, 12(4):800–806, 1975.
- [12] Harry Kesten. Subdiffusive behavior of random walk on a random cluster. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(4):425–487, 1986.
- [13] Christian Krattenthaler. Permutations with restricted patterns and Dyck paths. *Advances in Applied Mathematics*, 27(2):510–530, 2001.
- [14] Russell Lyons, Robin Pemantle, and Yuval Peres. Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab.*, 23(3):1125–1138, 1995.

- [15] Neal Madras and Lerna Pehlivan. Structure of random 312-avoiding permutations. *Random Structures & Algorithms*, 2015.
- [16] Sam Miner and Igor Pak. The shape of random pattern-avoiding permutations. *Advances in Applied Mathematics*, 55:86–130, 2014.
- [17] J. Neveu. Arbres et processus de Galton-Watson. *Ann. Inst. H. Poincaré Probab. Statist.*, 22(2):199–207, 1986.
- [18] Aaron Robertson, Dan Saracino, and Doron Zeilberger. Refined restricted permutations. *Annals of Combinatorics*, 6(3-4):427–444, 2002.